ECS455 2011/2, Chapter 3 Part 1, Dr.Prapun

call duration ~ $\mathcal{E}(\mu)$

In this note, we will look at the analysis of the call blocking probability in a cellular system under the M/M/m/m assumption. We assume

Poisson arrivals/requests with rate λ

(a) Blocked calls cleared
   - No queuing for call requests.
   - For every user who requests service, there is no setup time and the user is given immediate access to a channel if one is available.
   - If no channels are available, the requesting user is blocked without access and is free to try again later.

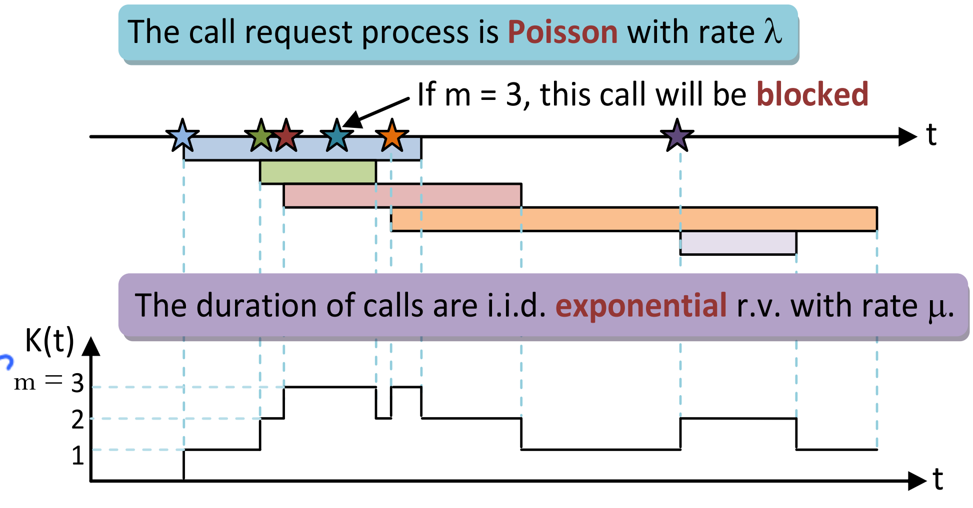
(b) Calls arrive as determined by a ***Poisson process***.

(c) Arrivals of requests are memoryless: all users, including blocked users, may request a channel at any time.

(d) There are an infinite number of users (with finite overall request rate).
   - The finite user results always predict a smaller likelihood of blocking. So, assuming infinite number of users provides a conservative estimate.

(e) The duration of the time that a user occupies a channel is ***exponentially*** distributed, so that longer calls are less likely to occur.

(f) There are m channels available in the trunking pool.
   - For us, m = the number of channels for a cell or for a sector.



State of the system.

Figure 1: M/M/m/m Assumption

Some of the conditions above are drawn in Figure 1. Later on, we will try to relax some of the assumptions above to make the analysis more realistic. In Figure 1, we also show one important parameter of the system: $K(t)$. This is the number of used channels at time t. When $K(t) < m$, new call can be made. When $K(t) = m$, new call request(s) will be blocked. So, we can find the call blocking probability by looking at the value of $K(t)$. In particular, we want to find out the proportion of time the system has $K = m$.

Poisson process and some probability concepts will be reviewed in Section 1. Most of the probability reviews will be put in footnotes so that they do not interfere with the flow of the presentation.

1 Poisson Process ← call generation/initiation process

In this section, we consider an important random process called ***Poisson process*** (PP). This process is a popular model for customer arrivals or calls requested to telephone systems.

1.1. We start by picturing a Poisson Process as a random arrangement of "marks" (denoted by "x") on the time axis. These marks may indicate the time that customers arrive or the time that call requests are made:

In the language of "queuing theory", the marks denote ***arrival times***.

1.2. In this class, we will focus on one kind of Poisson process called ***homogeneous Poisson process***. Therefore, from now on, when we say "Poisson process", what we mean is "homogeneous Poisson process".

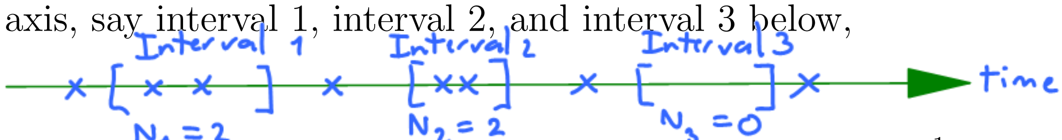
1.3. The first property of Poisson process that you should remember is that

> there is only one parameter for Poisson process.

This parameter is the ***rate*** or ***intensity*** of arrivals (the average number of arrivals per unit time.

- We used λ to denote this parameter.
- For homogeneous Poisson process, λ is a constant.
- For non-homogeneous Poisson process, λ is a function of time, say $\lambda(t)$
- Our λ is constant because we focus on homogeneous Poisson process.

1.4. How can λ, which is the only parameter, controls Poisson process? The key idea is that the Poisson process is as random/unstructured as a process can be. Therefore, if we consider many non-overlapping intervals on the time axis, say interval 1, interval 2, and interval 3 below,



and count the number of arrivals N_1, N_2 and N_3 in these intervals[1].

[1] Note that the numbers N_1, N_2, and N_3 are random. Because they are counting the number of arrivals, we know that they can be any non-negative integers:

$$0, 1, 2, 3, \ldots.$$

Because we don't know their exact values, we describe them via the likelihood or probability that they will take one of these possible values. For example, for N_1, we describe it by

$$P[N_1 = 0], P[N_1 = 1], P[N_1 = 2], \ldots$$

where $P[N_1 = k]$ is the probability that N_1 takes the value k. Such list of numbers is a bit tedious. So, we define a function

$$p_{N_1}(k) = P[N_1 = k].$$

This function $p_{N_1}(\cdot)$ tells the probability that N_1 will take a particular value (k). We call p_{N_1} the probability mass function (pmf) of N_1. At this point, we don't know much about $p_{N_1}(k)$ except that its values will be between 0 and 1 and that

$$\sum_{k=0}^{\infty} p_{N_1}(k) = 1.$$

These two properties are the necessary and sufficient conditions for any pmf.

Then, the numbers N_1, N_2 and N_3 in our example above should be independent[2]; for example, knowing the value of N_1 does not tell us anything at all about what N_2 and N_3 will be. This is what we are going to take as a vague definition of the "complete randomness" of the Poisson process.

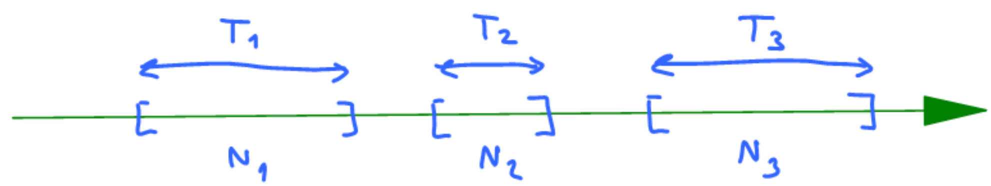
To summarize, now we have one more property of a Poisson process:

> The number of arrivals in non-overlapping intervals are independent.

1.5. Do we know anything else about N_1, N_2, and N_3? Again, we have only one parameter λ for a Poisson process. So, can we connect λ with N_1, N_2, and N_3?

Recall that λ is the average number of arrivals per unit time. So, if $\lambda = 5$ arrivals/hour, then we expect that N_1, N_2, and N_3 should conform with this λ, statistically.

Let's first be more specific about the time duration of the intervals that we have earlier. Suppose their lengths are T_1, T_2, and T_3 respectively



Then, you should expect[3] that

$$\begin{aligned} \mathbb{E}N_1 &= \lambda T_1, \\ \mathbb{E}N_2 &= \lambda T_2, \quad \text{and} \\ \mathbb{E}N_3 &= \lambda T_3. \end{aligned}$$

[2]By saying that something are independent, we mean it in terms of probability. In particular, when we say that N_1 and N_2 are independent, it means that

$$P\left[N_1 = k \text{ and } N_2 = m\right]$$

(which is the probability that $N_1 = k$ and $N_2 = m$) can be written as the product

$$p_{N_1}(k) \times p_{N_2}(k)$$

[3]Recall that $\mathbb{E}N_1$ is the expectation (average) of the random variable N_1. Formula-wise, we can calculate $\mathbb{E}N_1$ from

$$\mathbb{E}N_1 = \sum_{k=0}^{\infty} k \times P\left[N_1 = k\right];$$

that is the sum of the possible values of N_1 weighted by the corresponding propabilities

For example, suppose $\lambda = 5$ arrivals/hour and $T_1 = 2$ hour. Then you would see about $\lambda \times T_1 = 10$ arrivals during the first interval. Of course, the number of arrivals is random. SO, this number 10 is an average or the expected number, not the actual value.

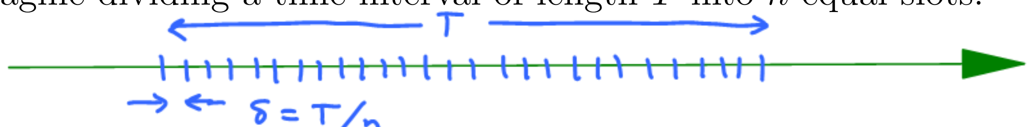
To summarize, we now know one more property of a Poisson process:

> For any interval of length T, the expected number of arrivals in this interval is given by
>
> $$\mathbb{E}N = \lambda T. \tag{1}$$

1.1 Discrete-time (small-slot) approximation of a Poisson process

1.6. The next key idea is to consider a small interval:

Imagine dividing a time interval of length T into n equal slots.



Then each slot would be a time interval of duration $\delta = T/n$. For example, if $T = 20$ hours and $n = 10,000$, then each slot would have length

$$\delta = \frac{T}{n} = \frac{20}{10,000} = 0.002 \text{ hour}.$$

Why do we consider small interval? The key idea is that as the interval becomes very small, then it is extremely unlikely that there will be more than 1 arrivals during this small amount of time. This statement becomes more accurate as we increase the value of n which decreases the length of each interval ever further. What we are doing here is an approximation of a continuous-time process by a discrete-time process.[4][5]

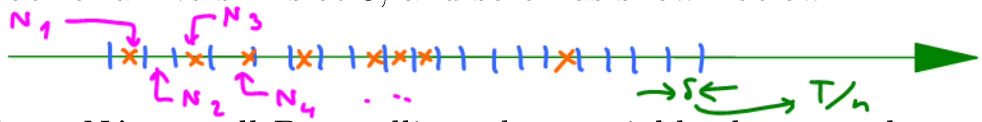
To summarize, we will consider the discrete-time approximation of the (continuous-time) Poisson process. In such approximation, the time axis is divided into many small time intervals (which we call "slots").

> When the interval is small enough, we can assume that at most 1 arrival occurs.

[4]You also do this when you plot a graph of any function $f(x)$. You divide the $x-$axis by many (equally spaced) values of x and then evaluate the values of the function at these values of x. You need to make sure that the values of x used are "dense" enough such that no surprising change in the function f is overlooked.

[5]If we wnat to be rigorous, we would have to bound the error from such approximation and show that the error disappear as $n \to \infty$. We will not do that here.

1.7. Let's look at the small slots more closely. Here, we let N_1 be the number of arrivals in slot 1, N_2 be the number of arrivals in slot 2, N_3 be the number of arrivals in slot 3, and so on as shown below.



Then, these N_i's are all Bernoulli random variables because they can only take the values 0 or 1. In which case, for their pmfs, we only need to specify one value $P[N_i = 1]$. Of course, knowing this, we can calculate $P[N_i = 0]$ by $P[N_i = 0] = 1 - P[N_i = 1]$.

Recall that the average $\mathbb{E}X$ of any Bernoulli random variable X is simply $P[X = 1]$.[6] So, if we know $\mathbb{E}X$ for Bernoulli random variable, then we know right away that $P[X = 1] = \mathbb{E}X$ and $P[X = 0] = 1 - \mathbb{E}X$.

Now, it's time to use what we learned about Poisson process. The slots that we consider before are of length T/n. So, the random variables $N_1, N_2, N_3, \ldots$ share the same expected value

$= P[N_3 = 1]$
$= P[N_2 = 1]$

$$\mathbb{E}N_1 = \mathbb{E}N_2 = \mathbb{E}N_3 = \cdots = \lambda\delta. \quad = P[N_1 = 1]$$

For example, with $\lambda = 5$, $T = 20$, and $N = 10,000$, the expected number of arrivals in a slot is

$$\lambda\delta = \lambda\frac{T}{n} = 0.01 \text{ arrivals.}$$

Because these N_i's are all Bernoulli random variables and because they share the same expected value, we can conclude that they are identically distributed; that is their pmf's are all the same. Furthermore, because the slots do not overlap, we also know that the N_i's are independent. Therefore,

> the N_i's are i.i.d. Bernoulli random variables whose pmf's are given by
>
> $$p_1 = P[N_i = 1] = \lambda\delta \quad \text{and} \quad p_0 = P[N_i = 0] = 1 - \lambda\delta,$$
>
> where δ is the length of each slot.

1.8. At this point, you can use `MATLAB` to generate a Poisson process with arrival rate λ using discrete-time approximation. Here are the steps:

[6] For Bernoulli random variable X, the average is

$$\mathbb{E}X = \cancel{0 \times P[X = 0]} + 1 \times P[X = 1] = P[X = 1].$$

For conciseness, we usually let $p_0 = P[X = 0]$ and $p_1 = P[X = 1]$. Hence, $\mathbb{E}X = p_1$.

(a) First, we fix the length T of the whole simulation. (For example, $T = 20$ hours.)

(b) Then, we divide T into n slots. (For example, $n = 10,000$.)

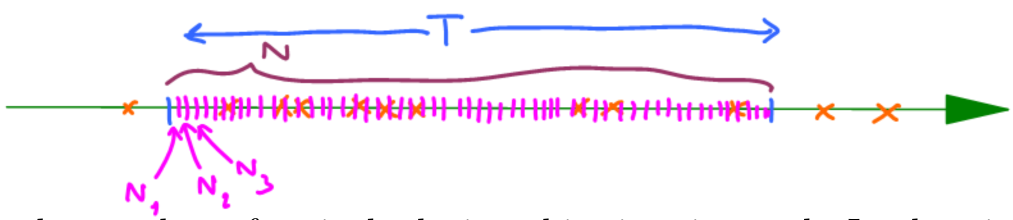
(c) For each slot, only two cases can happen: 1 arrival or no arrival. So, we generate Bernoulli random variable for each slot with $p_1 = \lambda \times T/n$. (For example, if $\lambda = 5$ arrival/hr, then $p_1 = 0.01$.)

To do this for n slots, we can use the command `rand(1,n) < p1` or `binornd(1,p1,1,n)`.

1.9. Note that what we have just generated is exactly ***Bernoulli trials*** whose success probability for each trial is $p_1 = \lambda\delta$. In other words, a Poisson process can be approximated by Bernoulli trials with success probability $p_1 = \lambda\delta$.

1.2 Properties of Poisson Processes

1.10. What we want to do next is to revisit the description of the number of arrivals in a time interval. Now, we will NOT assume that length of the time interval is short. In particular, let's reconsider an interval of length T below.



Let N be the number of arrivals during this time interval. In the picture above, $N = 4$.

Again, we will start with a discrete-time approximation; we divide T into n small slots of length $\delta = \frac{T}{n}$. In the previous subsection, we know that the number of arrivals in these intervals, denoted by $N_1, N_2, \ldots, N_n$ can be well-approximated by i.i.d. Bernoulli with probability of having exactly one arrival $= \lambda\delta$. (Of course, we need δ to be small for the approximation to be precise.) The total number of arrivals during the original interval of length T can be found by summing the values of the N_i's:

$$N \approx N_1 + N_2 + \cdots + N_n. \tag{2}$$

You may recall, from introductory probability class, that

(a) summation of n Bernoulli random variables with success probability p gives a binomial(n, p) random variable[7]

and that

(b) the binomial(n, p) random variable whose n is large and p is small can be well approximated by a Poisson random variable with parameter $\alpha = np$ [8]

Therefore, the pmf of the random variable N in (2) can be approximated by a Poisson pmf whose parameter is

$$\alpha = np_1 = n\lambda \frac{T}{n} = \lambda T.$$

This approximation gets more precise when n is large (δ is small). In fact, in the limit as $n \to \infty$ (and hence $\delta \to 0$), the random variable N is $\mathcal{P}(\lambda T)$. Recall that the expected value of $\mathcal{P}(\alpha)$ is α. Therefore, λT is the expected value of N. This agrees with what we have discussed before in (1).

In conclusion,

the number N of arrivals in an interval of length T is a Poisson random variable with mean (parameter) λT

1.11. Now, to sum up what we have learned so far, the following is one of the two main properties of a Poisson process

The number of arrivals $N_1, N_2, N_3, \ldots$ during non-overlapping time intervals are independent Poisson random variables with mean $\lambda \times$ the length of the corresponding interval.

1.12. Another main property of the Poisson process, which we will state without proof, is that

---

[7] X is a ***binomial*** random variable with size $n \in \mathbb{N}$ and parameter $p \in (0, 1)$ if

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, 2, \ldots, n\} \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

We write $X \sim \mathcal{B}(n, p)$ or $X \sim \text{binomial}(p)$. Observe that $\mathcal{B}(1, p)$ is Bernoulli with parameter p. Note also that $\mathbb{E}X = np$.

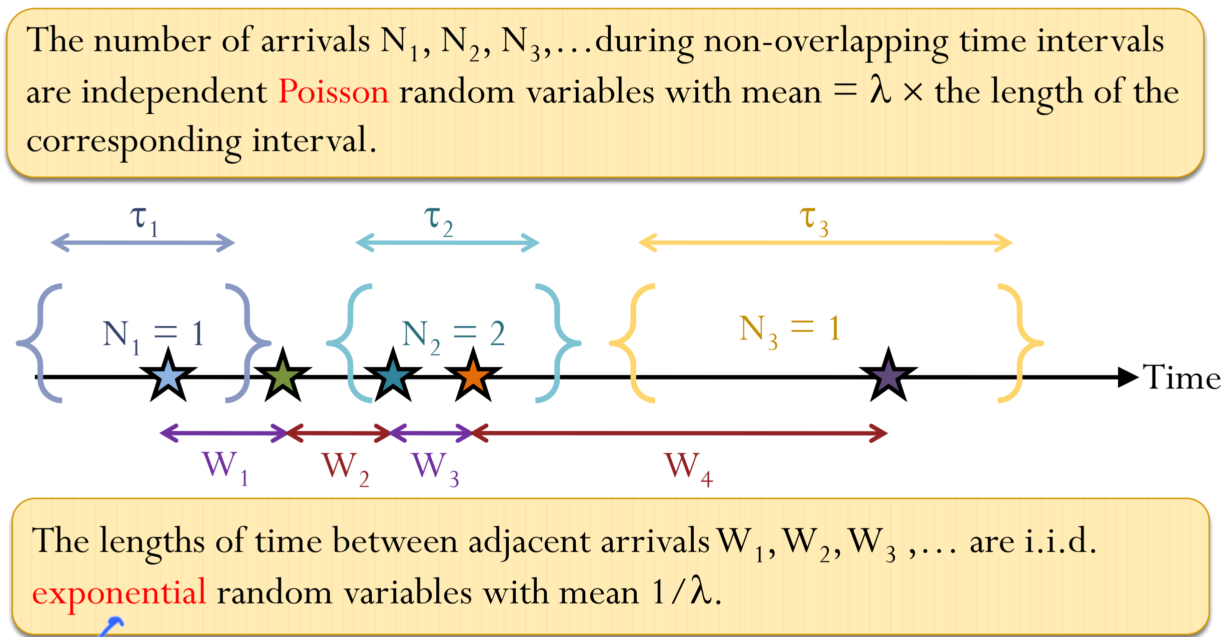
[8] X is a ***Poisson*** random variable with ***parameter*** $\alpha > 0$ if

$$p_X(k) = \begin{cases} e^{-\alpha} \frac{\alpha^k}{k!}, & k \in \{0, 1, 2, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

We write $X \sim \mathcal{P}(\alpha)$ or Poisson(α). Note also that $\mathbb{E}X = \alpha$.

The lengths of time between adjacent arrivals $W_1, W_2, W_3, \ldots$ are i.i.d. exponential[9] random variables with mean $1/\lambda$.

This property can be derived by looking at the discrete-time approximation of the Poisson process. In the discrete-time version, the time until the next arrival is geometric. In the limit, the geometric random variable becomes exponential random variable. Both main properties of Poisson process are shown in Figure 2. The small slot analysis (discrete-time approximation), which can be used to prove the two main properties, is shown in Figure 3.



$\mathcal{E}(\lambda)$

$\lambda e^{-\lambda w}, \; w > 0$

Figure 2: Two main properties of a Poisson process

2 Derivation of the Erlang B Formula

The Erlang B formula along with the definitions of the parameters used in it is shown in Figure 4.

[9]The exponential distribution is denoted by $\mathcal{E}(\lambda)$. An exponential random variable X is characterized by its probability density function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

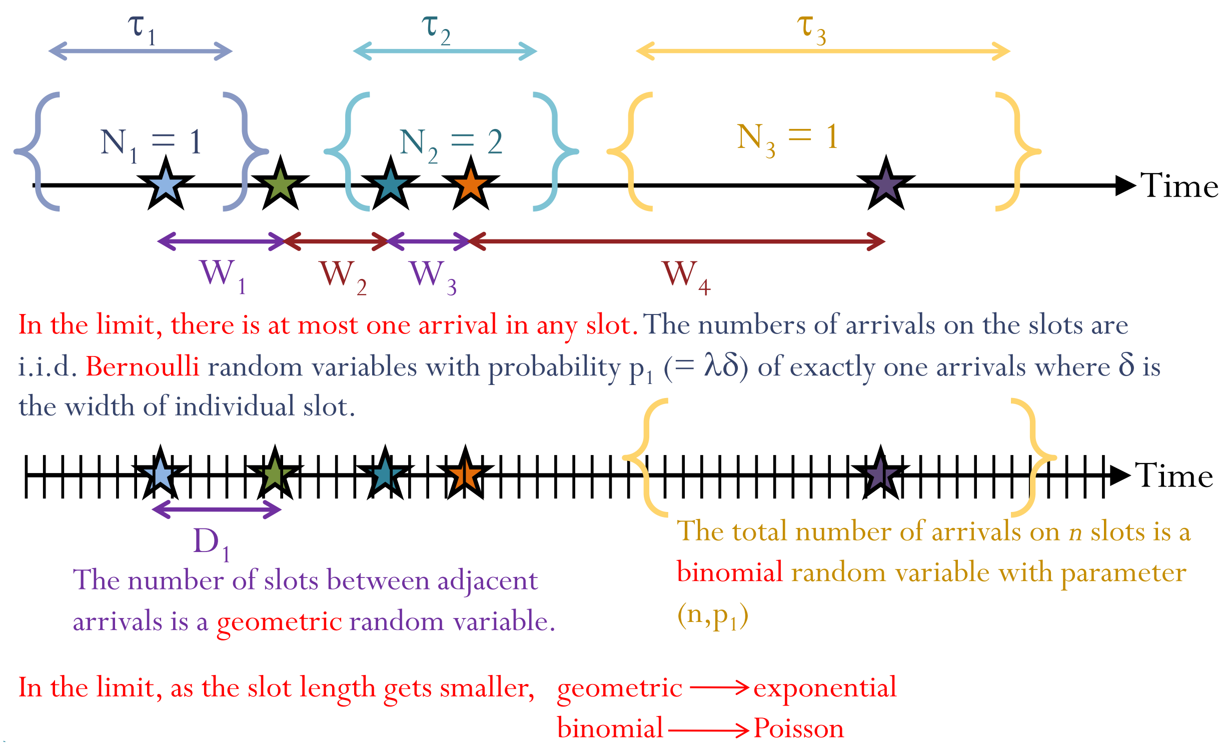


Figure 3: Small slot analysis (discrete-time approximation) of a Poisson process

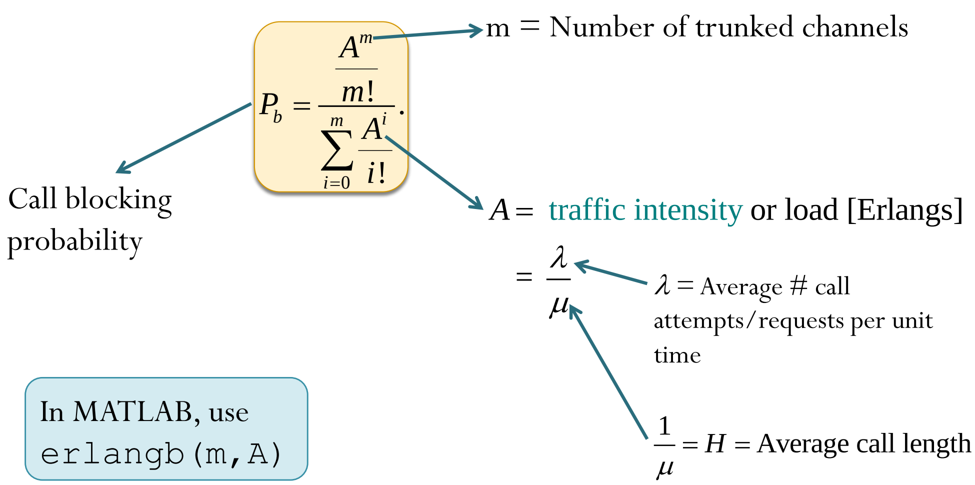


Figure 4: Erlang B Formula

call length or duration
$\sim \mathcal{E}(\mu)$